

A CHARACTERIZATION OF COBOUNDARY POISSON LIE GROUPS AND HOPF ALGEBRAS

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Abstract. We show that a Poisson Lie group (G, π) is coboundary if and only if the natural action of $G \times G$ on $M = G$ is a Poisson action for an appropriate Poisson structure on M (the structure turns out to be the well known π_+). We analyze the same condition in the context of Hopf algebras. Quantum analogue of the π_+ structure on $SU(N)$ is described in terms of generators and relations as an example.

1. Preliminaries. For the theory of Poisson Lie groups we refer to [1, 2, 3, 4, 5]. We follow the notation used in our previous papers [6, 7].

A *Poisson Lie group* is a Lie group G equipped with a Poisson structure π such that the multiplication map is Poisson. The latter property is equivalent to the following property (called *multiplicativity* of π)

$$(1) \quad \pi(gh) = \pi(g)h + g\pi(h) \quad \text{for } g, h \in G.$$

Here $\pi(g)h$ denotes the right translation of $\pi(g)$ by h etc. This notation will be used throughout the paper.

A Poisson Lie group is said to be *coboundary* if

$$(2) \quad \pi(g) = rg - gr$$

for a certain element $r \in \mathfrak{g} \wedge \mathfrak{g}$. Here \mathfrak{g} denotes the Lie algebra of G . Any bivector field of the form (2) is multiplicative. It is Poisson if and only if

$$[r, r] \in (\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g})_{\text{inv}}$$

(the Schouten bracket $[r, r]$ is \mathfrak{g} -invariant). In this case the element r is said to be a *classical r -matrix* (on \mathfrak{g}).

For any Poisson Lie group (G, π) , the antipode map $g \mapsto Sg := g^{-1}$ is anti-Poisson:

$$(3) \quad S_*\pi = -\pi.$$

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2. Gauge transformations of a lattice connection on one link. Consider the following action

$$(4) \quad (G \times G) \times G \ni ((g_0, g_1), g) \mapsto g_1 g g_0^{-1} \in G$$

of $G \times G$ on G . This type of action is familiar in gauge field theory on the lattice. We think here about an ‘elementary’ lattice composed of only one link with two ends: 0 and 1. Elements g_0 and g_1 are the values of the gauge transformation at the lattice sites 0 and 1, respectively. The connection on the link is represented by the element g .

One can ask if it is possible to consider the gauge group to be a Poisson Lie group (or, a quantum group). In this case it is natural to require the action (4) to be a Poisson action (i.e. the map (4) to be a Poisson map).

DEFINITION 1. A Poisson Lie group (G, π) is said to be *gauge-admissible* if there exists a Poisson structure ρ on G such that the map (4) is a Poisson map as a map from $(G, \pi) \times (G, \pi) \times (G, \rho)$ to (G, ρ) .

Note that we treat the gauge group differently than the space of connections (even if the latter is parameterized by the group manifold).

PROPOSITION 1. *A Poisson Lie group is gauge admissible if and only if it is coboundary.*

Proof. Let (G, π) be a Poisson Lie group. It is gauge admissible if and only if the map

$$(5) \quad G \times G \times G \ni (x, y, z) \mapsto xyz^{-1} \in G$$

is Poisson as a map from $(G, \pi) \times (G, \rho) \times (G, \pi)$ to (G, ρ) or, equivalently (using (3)), if the map $\Psi: G \times G \times G \rightarrow G$ defined by

$$\Psi(x, y, z) = xyz$$

is Poisson as a map from $(G, \pi) \times (G, \rho) \times (G, -\pi)$ to (G, ρ) . By a similar reasoning which leads to (1), this is equivalent to

$$(6) \quad \rho(xyz) = \pi(x)yz + x\rho(y)z - xy\pi(z) \quad \text{for } x, y, z \in G.$$

We have two following particular cases of this equality. If we set $z = e$ (the group unit), we get

$$(7) \quad \rho(xy) = \pi(x)y + x\rho(y),$$

and if we set $x = e$, we get

$$(8) \quad \rho(yz) = \rho(y)z - y\pi(z).$$

It is easy to see that (7) and (8) together are equivalent to (6). Since $\rho = \pi$ is a particular solution of (7), the general solution of (7) is given by

$$(9) \quad \rho(g) = \pi(g) + gA,$$

where $A \in \mathfrak{g} \wedge \mathfrak{g}$. Since $\rho = -\pi$ is a particular solution of (8), the general solution of (8) is given by

$$(10) \quad \rho(g) = -\pi(g) + Bg,$$

where $B \in \mathfrak{g} \wedge \mathfrak{g}$. For the compatibility of (9) and (10) we must have

$$\pi(g) = \frac{Bg - gA}{2}.$$

Since $\pi(e) = 0$, we have $B = A$, and finally

$$\pi(g) = \frac{Ag - gA}{2}, \quad \rho(g) = \frac{Ag + gA}{2}.$$

This shows that (G, π) is gauge-admissible if and only if it is coboundary (with $r = A/2$; note that if r is the classical r -matrix then $\pi_+(g) := rg + gr = \rho(g)$ is automatically a Poisson bivector field). ■

It is clear that for a given coboundary Poisson structure π , all possible ρ are obtained from one by adding an invariant element of $\mathfrak{g} \wedge \mathfrak{g}$. In particular, if \mathfrak{g} is semisimple, then ρ is unique.

3. Hopf algebra case.

Let (H, m, Δ) be a Hopf algebra. Here $m: H \otimes H \rightarrow H$ and $\Delta: H \rightarrow H \otimes H$ denote the multiplication and the comultiplication in H . Let I and c denote the unit and counit of the Hopf algebra.

We set

$$\Psi := m(m \otimes \text{id}) = m(\text{id} \otimes m)$$

and ask when there exists a (new) coalgebra structure $\tilde{\Delta}$ (with the same counit c) on H such that Ψ is a morphism from $(H, \Delta) \otimes (H, \tilde{\Delta}) \otimes (H, \Delta^{\text{op}})$ to $(H, \tilde{\Delta})$. Here Δ^{op} is the comultiplication opposite to Δ : $\Delta^{\text{op}} = P \circ \Delta$, where P is the permutation in the tensor product.

The condition for Ψ to be such a morphism reads:

$$(11) \quad \tilde{\Delta}\Psi = (\Psi \otimes \Psi)(\text{id} \otimes \text{id} \otimes P \otimes \text{id} \otimes \text{id})(\text{id} \otimes P \otimes P \otimes \text{id})(\Delta \otimes \tilde{\Delta} \otimes \Delta^{\text{op}}),$$

and is equivalent to two following conditions

$$(12) \quad \tilde{\Delta}m = (m \otimes m)(\text{id} \otimes P \otimes \text{id})(\Delta \otimes \tilde{\Delta})$$

$$(13) \quad \tilde{\Delta}m = (m \otimes m)(\text{id} \otimes P \otimes \text{id})(\tilde{\Delta} \otimes \Delta^{\text{op}})$$

(they follow from (11) by applying it to $\text{id} \otimes \text{id} \otimes I$ and $I \otimes \text{id} \otimes \text{id}$, respectively). It is easy to solve these conditions for $\tilde{\Delta}$. Applying (12) to $\text{id} \otimes I$, we get

$$(14) \quad \tilde{\Delta}(a) = \Delta(a) \mathbf{R} \quad a \in H,$$

where the multiplication is that of $H \otimes H$ and

$$\mathbf{R} := \tilde{\Delta}(I).$$

It is easy to see that (14) solves (12) for any \mathbf{R} .

Similarly, applying (13) to $I \otimes \text{id}$, we get

$$\tilde{\Delta}(a) = \mathbf{R} \Delta^{\text{op}}(a) \quad a \in H.$$

This is a solution of (13) for any \mathbf{R} . It follows that the general solution of (11) is (14), where the R -matrix \mathbf{R} satisfies the compatibility condition

$$(15) \quad \Delta(a) \mathbf{R} = \mathbf{R} \Delta^{\text{op}}(a) \quad a \in H.$$

It is easy to see that $\tilde{\Delta}$ is coassociative if and only if

$$(16) \quad [(\Delta \otimes \text{id}) \mathbf{R}](\mathbf{R} \otimes I) = [(\text{id} \otimes \Delta) \mathbf{R}](I \otimes \mathbf{R}).$$

Indeed,

$$(\tilde{\Delta} \otimes \text{id})\tilde{\Delta}(a) = [(\Delta \otimes \text{id})(\Delta(a)\mathbf{R})](\mathbf{R} \otimes \text{id}) = [(\Delta \otimes \text{id})\Delta(a)][(\Delta \otimes \text{id})\mathbf{R}](\mathbf{R} \otimes \text{id}),$$

$$(\text{id} \otimes \tilde{\Delta})\tilde{\Delta}(a) = [(\text{id} \otimes \Delta)(\Delta(a)\mathbf{R})](\text{id} \otimes \mathbf{R}) = [(\text{id} \otimes \Delta)\Delta(a)][(\text{id} \otimes \Delta)\mathbf{R}](\text{id} \otimes \mathbf{R}).$$

Concluding: the question at the beginning of this section has an affirmative answer if and only if there exists an element $\mathbf{R} \in H \otimes H$ such that (15), (16) hold and

$$(c \otimes \text{id})\mathbf{R} = I = (\text{id} \otimes c)\mathbf{R}.$$

A Hopf algebra satisfying those conditions might be called *gauge-admissible* or *coboundary*. (I do not know whether one can always choose \mathbf{R} to be ‘unitary’, like in [2]: $\mathbf{R}_{12}\mathbf{R}_{21} = I \otimes I$).

The Hopf algebra considered in this section should be interpreted as a dual of the Hopf algebra of functions on a quantum group (quantized universal enveloping algebra). In the next section we give an example of a ‘gauge-admissible’ matrix quantum group.

4. Example in terms of generators and relations.

Let

$$(17) \quad R(u \oplus u) = (u \oplus u)R$$

be a part of relations defining a matrix quantum group (A, u) . Here $u = (u_{ij})_{i,j=1,\dots,n}$ is the defining representation of the quantum group, R is the fundamental intertwiner (R -matrix of FRT-type) and we use the Woronowicz’s notation for the ‘matrix’ tensor product. Let us note that we have

$$(18) \quad \tilde{R}(u^{-1} \oplus u^{-1}) = (u^{-1} \oplus u^{-1})\tilde{R},$$

where $\tilde{R} := PRP$. Let us denote by B the algebra generated by the entries of the $n \times n$ matrix w and relations

$$(19) \quad R(w \oplus w) = (w \oplus w)\tilde{R}.$$

It is easy to see that there exists exactly one homomorphism \mathfrak{h} (quantum gauge transformation – the analogue of (5)) from B to $A \otimes B \otimes A$ such that

$$\mathfrak{h}(w^i_j) = \sum_{kl} u^i_k \otimes w^k_l \otimes (u^{-1})^l_j,$$

or, using the Woronowicz’s notation,

$$(\mathfrak{h} \otimes \text{id})(w) = u \oplus w \oplus u^{-1}$$

(here w is understood as an element of $\mathbf{End}(\mathbb{C}^n) \otimes B$). In order to see that $u \oplus w \oplus u^{-1}$ satisfies the same relations as w , we notice that

$$(u \oplus w \oplus u^{-1}) \oplus (u \oplus w \oplus u^{-1}) = (u \oplus u) \oplus (w \oplus w) \oplus (u^{-1} \oplus u^{-1})$$

and use subsequently (17), (18) and (19).

In order to be more precise, we consider now a specific matrix quantum group, namely $SU_q(n)$, as given in [8]. The $*$ -algebra A of ‘regular functions’ on $SU_q(n)$ is the one generated by the entries of an $n \times n$ matrix u and the following relations:

$$(20) \quad u^{(n)}E = E, \quad E'u^{(n)} = E', \quad uu^* = I_n \otimes I_A = u^*u.$$

Here $u^{(n)}$ is the n -th tensor power of u , E is the ‘ q -deformed’ volume element

$$E^{i_1 i_2 \dots i_n} = (-q)^{\text{number of inversions in } (i_1, \dots, i_n)}, \quad E'_{i_1 \dots i_n} = E^{i_1 \dots i_n}$$

(for $(i_1 \dots i_n)$ not being a permutation we set $E^{i_1 \dots i_n} = 0$) and I_n is the unit $n \times n$ matrix. Note that in this case

$$(u^{-1})^{(n)} \tilde{E} = tE, \quad \tilde{E}'(u^{-1})^{(n)} = \tilde{E}',$$

where

$$\tilde{E} = P_{\text{total}} E, \quad \tilde{E}' = E' P_{\text{total}},$$

P_{total} being the total permutation $(1, 2, \dots, n) \mapsto (n, \dots, 2, 1)$.

Let B be the $*$ -algebra generated by the entries of an $n \times n$ matrix w and relations

$$(21) \quad w^{(n)} \tilde{E} = (-1)^{\frac{n(n-1)}{2}} E, \quad E' w^{(n)} = (-1)^{\frac{n(n-1)}{2}} \tilde{E}', \quad ww^* = I_n \otimes I_B = w^* w.$$

It is easy to check that $u \oplus w \oplus u^{-1}$ satisfies the same relations, hence we have the ‘gauge transformations’ on the quantum level.

It is essential to know if algebra B has a correct size (Poincaré series), i.e. if the deformation is *flat*. We shall show that B is actually isomorphic to A . To this end, consider the change of variables

$$u = \varepsilon w P_{\text{total}}$$

in (20), where ε is such a complex number that $\varepsilon^n = (-1)^{\frac{n(n-1)}{2}} = \det P_{\text{total}}$. It is easy to see that relations (20) are now transformed to relations (21).

5. Remarks.

5.1. The algebra B defined in (21) is the quantum counterpart of the Poisson structure $\pi_+(g) = rg + gr$ on $SU(n)$. The case of a general group is sketched in (19). Note that if we substitute $u = wg_0$ in (17) where g_0 is an element of the classical group such that

$$(22) \quad (g_0 \otimes g_0) R(g_0^{-1} \otimes g_0^{-1}) = PRP,$$

then we obtain relations (19). One can check that the well known R -matrix for the A_n series satisfies (22) if we choose $g_0 = \varepsilon P_{\text{total}}$. The corresponding fact for Poisson groups means that we find an element $g_0 \in G$ such that

$$(23) \quad \pi = \pi_+ g_0$$

i.e. $\pi(gg_0) = \pi_+(g)g_0$, that is to say

$$rgg_0 - gg_0r = rgg_0 + grg_0,$$

or,

$$g_0 r g_0^{-1} = -r,$$

or,

$$(24) \quad \pi_+(g_0) = 0.$$

For instance in the case of the standard r -matrix of the A_n -series,

$$r = \sum_{j < k} e_j^k \wedge e_k^j,$$

$g_0 := \varepsilon P_{\text{total}}$ will do the job, because $P e_j = e_{j'}, j' := n + 1 - j$.

5.2. Formula (14) was used in [9] to discuss twisting Hopf algebras by 2-cocycles. The Poisson structure π_+ is isomorphic to π by a translation (23) if and only if it vanishes at some point (namely g_0 , see (24)). This situation (and previously discussed isomorphism of B with A) corresponds to twisting by a coboundary.

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